Value of information sharing in marine mutual insurance

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Abstract

With empirical evidence from marine mutual insurance (MMI), an impulse feedback model is constructed to address how information sharing can help increase both the social welfare and the efficiency of operation of the MMI system. Focusing on information sharing, this paper considers the premium policy optimization of a pure (i.e., non-stock consideration) mutual insurance system with a homogeneous market of identical members. Our findings confirm that the principle of information sharing can be attained under equal-risk pooling, but not necessarily under unequal-risk pooling, and reveal that quantifiable differences exist in the valuation of information sharing under the two schemes of risk pooling. It indicates that the key to successful MMI is equal-risk pooling. Algorithms are developed to compute the value of information sharing by solving the HJB equations and quasi-variational inequalities. It determines that information sharing can achieve best social welfare as well as efficient operation of a P&I Club. The conclusion provides a scientific basis for both managerial strategy and competition regulation. The findings are also applicable to a wide range of reserve and inventory management problems.

Key words: Information sharing; Mutual Insurance; P&I Club

1. Introduction

In the marine insurance market, mutual Clubs are not the pure commercial insurance enterprises as conventionally defined. Insurance is based on membership of the Club. An individual ship-owner, who wants to pool his risk in a mutual Club, must first obtain membership through payment of a membership fee, which is deemed to be the premium as defined in commercial insurance (Cass, Chichilnisky and Wu, 1996; Lamm-Tennant and Starks, 1993).

Numerous studies into the formation of mutual insurers concern adverse selection (Smith and Stutzer, 1990; Ligon and Thistle, 2005), moral hazards (Smith and Stutzer, 1995) and information asymmetry (Cabrales et al, 2003). At the outset of the insurance industry, there were only mutual insurers, and because of the existence of adverse selection and information asymmetry, the original risk pool degenerated into two types of sub-risk-pools formed by the homogeneous assureds.

There are a total 28 Marine Mutual Clubs, 13 of which are members of the International Group of P&I Clubs (see table 1). The 13 P&I Clubs that comprise the International Group of P&I Clubs (the IG) are mutual not-for-profit insurance organizations that between them cover third party liabilities, which include pollution, loss of life and personal injury, cargo loss and damage and collision risks. The Clubs are mutual organizations, that is, the shipowner members are both insured and insurers, as the members both own and control their individual clubs (Gold, 2002; Hazelwood, 2000). The day-to-day activities and operations of the Clubs are delegated to managers. The 13 P&I Clubs cover about 95% of the world's ocean-going vessels in terms of tonnage (JLT, 2003; Golish, 2005). This monopoly position, and the way that P&I Clubs operate, have triggered two European Commission (EC) investigations and rulings, in 1985 and 1999 respectively, in the field of competition law (Gyselen, 1999; EC, 1985, 1999).

	Name of Club	Year Established	Headquarters	IG member	Annual Growth	Size (gt) (million)
1	UK	1869	London	Yes	10.06%	120
2	Gard	1907	Arendal	Yes	4.76%	98
3	Britannia	1854	London	Yes	5.68%	80
4	Steamship	1974	London	Yes	1.38%	65
5	Standard	1885	London	Yes	13.03%	58
6	Japan	1950	Tokyo	Yes	2.24%	54
7	Skuld	1897	Oslo	Yes	-5.23%	52
8	W. England	1856	London	Yes	3.73%	46
9	N. England	1860	Newcastle	Yes	19.72%	43
10	London	1866	London	Yes	2.46%	28
11	American	1917	New York	Yes	19.33%	18
12	Swedish	1872	Goteborg	Yes	2.36%	15
13	China	1984	Beijing	No	30.47%	9
14	Shipowners	1855	London	Yes	9.86%	9
	Average				**9.11%**	**46**

Table 1: Major Marine Mutual Insurance

Notes: 1. This has been combined from various sources by the authors.
2. IG – International Group of P&I Clubs; g.t. – gross tonnage of entered ships.

A key message from the EC is that the P&I Club system should be more transparent both to its members and to the outside world, in order to ensure full implementation of the principle of information sharing (Garner, 1999; Macey, 2004). The International Maritime Organization (IMO) has also asked P&I Clubs to open their books, for the general interest of "safer ships and cleaner oceans." Such suggestion has always been rejected by P&I Clubs under the defense of 'protection of privacy', so the lack of data from P&I Clubs contributes to the need for academic research to be conducted in this area (Johnstad, 2000; Bennett, 2000). Much information about MMI in general has been "veiled in antiquity and lost in obscurity" (Dover, 1975). All of these discussions have pointed to one critical legal argument — whether MMI should open up or continue to keep closed information on claims records and other information. However, this paper determines that information sharing can achieve best social welfare as well as efficient operation of a P&I Club.

Assuming the standard Brownian motion characterization of a claim process associated with each individual vessel of a club (Tapiero and Jacque, 1987; Asmussen and Taksar, 1997; Siegl and Tichy, 1999), we develop for the MMI cost minimization problem an impulse feedback model of a pure MMI (Bensoussan and Lions, 1984; and Aubin, 2000). We focus the analysis on a comparison of optimal insurance positions (i.e., funds on hand) under two information structures — unequal-risk versus equal-risk. In practice, P&I Clubs collect premiums at the beginning of each policy year on 28 February. Therefore, we model a P&I Club as an impulse control system in the sense that the total reserve of the Club is "reset" by an impulse premium control, so that insurance claims incurred during the policy year can be sufficiently covered at a desirable level of risk pooling.

Through an impulse feedback analysis, we first calculate the optimal premium policies (in terms of total-cost minimization) under different risk pooling structures (unequal- versus equal-risk pooling). We then determined a quantifiable extra value in an unequal-risk pooling MMI system, as compared with an equal-risk pooling one. In this paper, this extra value term is referred to as the "value" of information sharing. We develop algorithms to compute the value of information sharing, by solving the HJB equations and quasi-variational inequalities.

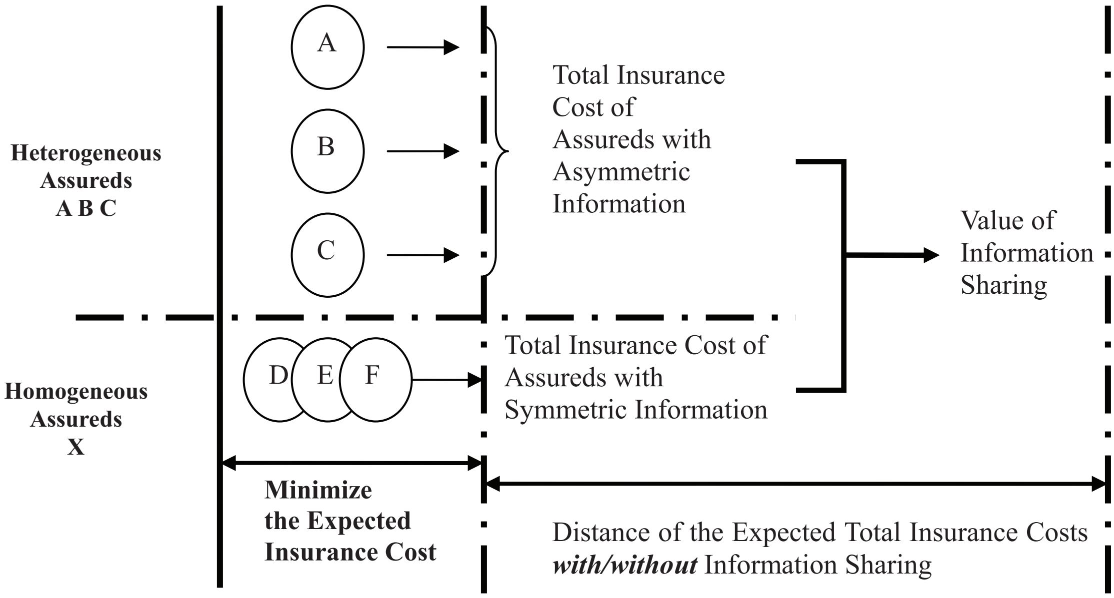

Figure 1: Value of Information Sharing

2. Impulse Feedback Model for Unequal-Risk Pooling

Consider a general mutual insurance Club of *n* members over a time horizon $[0,T]$ (when $T=\infty$, $[0,T]=[0,\infty)\bigcup\{\infty\}$), during which each member *i* establishes an account of insurance position y_t^i (i.e., individual account balance) at time *t*, and a record of the individual claims incurred by member *i* (denoted by x_t^i). The records of individual accounts and claims are kept confidential at the Club. Let each Gaussian claim process $x_t^i \sim N(\lambda_i,\sigma_i^2)$ be characterized as a drifted Brownian motion, as follows:

$$dx_t^i = \lambda_i dt + \sigma_i d\widetilde{w}_t^i \quad (i=1,\cdots,n), \tag{1}$$

where $d\widetilde{w}_t^i$ represents a Wiener differential. The Club will review the current insurance position and claims outlook at the beginning of each renewal period k (e.g., each year *k*, $k=1,2,\cdots$). Following the review, the Club then determines for each member a premium call (an impulse control), denoted as q_k^i, so as to "reset" the member's insurance position; i.e., $y_k^i = y_{k-1}^i + \int_{k-1}^{k} dy_t^i + q_k^i$, where dy_t^i is the differential of the insurance position during a review period $t\in[k-1,k)$. We characterize this differential using the following differential characteristics:

$$\begin{aligned} dy_t^i &= h_i(y_t^i)dt - dx_t^i \\ &= \mu_i(y_t^i)dt + \sigma_i dw_t^i \ (i=1,\cdots,n), \end{aligned} \tag{2}$$

where $h_i(y_t^i)$ represents the rate of individual position consumption (e.g., operational and management costs) excluding the claim coverage costs, $\mu_i(y_t^i) = h_i(y_t^i) - \lambda_i$, and $dw_t^i = -d\widetilde{w}_t^i$. For the rest of the paper, we adopt the following notation system: Denote by small letter the vector, e.g., the position vector as $y_t = (y_t^1, \cdots, y_t^n)$, and by a capital letter the sum $Y_t = \sum_{i=1}^{n} y_t^i$. The dynamics of an MMI system can be concisely presented in vector form as

$$dy_t = \mu(y_t)dt + \Theta dw_t ,$$

where $\mu(y_t)$ is an $n \times 1$ vector drift, $\Theta = (\theta_{ij})_{n \times n}$ is a matrix disturbance with $\theta_{ii} = \sigma_i$ and $\theta_{ij} = 0$ for $i \neq j$, and dw_t is an *n*-dimension Wiener differential (Gollier and Wibaut, 1992; Taksar, 2000). As a function of the Club's state y_t, let $L_i(y_t)$ be a nonnegative non-decreasing management cost (Lagrangian) associated with member *i* of the Club, and $L(y_t) = \sum_{i=1}^{n} L_i(y_t)$ be the total MI operating cost of the Club. Similarly, there is an impulse cost (including the premium payment) associated with each premium call q_k^i for renewal year *k*, denoted by $K_i(q_k)$, and an aggregate impulse cost for the entire Club $K(q_k) = \sum_{i=1}^{n} K_i(q_k^i)$ for all $k \in T_q$. Thus, the Club is facing the problem of minimizing the total insurance costs under unequal-risk information (Kavadias and Loch, 2003; Kulkarni, Magazine and Raturi, 2004), which we formulate, in vector form, as an impulse control system (Bensoussan and Lions, 1984; Aubin, 2000), as follows:

$$\varphi\,(y_0) = \min_{q_k} E\left(\sum_{k \in T_q} K(q_k)e^{-rk} + \int_0^T L(y_t)e^{-rt}dt + \Omega(y_T)e^{-rT} \right), \text{ for any } y_0 \qquad (3)$$

Subject to:

$$\begin{cases} dy_t = \mu\,(y_t)dt + \Theta dw_t \\ y_k = y_{k-1} + \int_{k-1}^{k} dy_t + q_k \end{cases}$$

where $\Omega(y_T)$ is a terminal function.

2.1. Optimal Premium Policy under Unequal-Risk: Impulse Feedback Control

According to impulse control theory, the value function $\varphi(y)$ is a solution to the following quasi-variational inequalities (QVI) system (see Polyanin and Zaitsev, 1995; Aubin, 2000; Liu, 2004):

$$\begin{cases} \text{i)}\quad r\varphi(y) - A\varphi(y) - H(y, D\varphi) \le 0 \\ \text{ii)}\ \ \varphi(y) - (\varphi * K)(y) \le 0 \\ \text{iii)}\ \big(\varphi - (\varphi * K)\big)\ \big((r\varphi - A\varphi) - H\big) = 0 \end{cases} \qquad (4)$$

where:

1) $D\varphi = (D_y\varphi) = (\varphi_{y^1}, \cdots, \varphi_{y^n})$ represents a gradient of φ, where φ_{y^i} is the marginal insurance cost shared by member *i*.
2) $A = \sum_{i=1}^{n} \frac{1}{2}\sigma_i^2 \frac{\partial^2}{\partial (y^i)^2}$ is a second-order differential (in viscosity sense) operator.
3) $H(y, D\varphi) = \langle D\varphi, \mu \rangle + L$ is the Hamiltonian of the MI system (3).
4) $(\varphi * K)(y) := \inf_q \big(\varphi(y+q) + K(q)\big)$ is termed the *inf-convolution* of functions φ and *K*.

According to impulse control analysis, there exists for system (3) an optimal position $y^* = (y^{1*}, \cdots, y^{n*})$ termed an impulse feedback control, to which the actual position at the beginning of each renewal period must be "reset" by collecting premiums accordingly. For the sake of reference, we present below the optimal impulse feedback policy for the individual positions, but with the proofs omitted (Aubin, 2000; Liu, 2004).

PROPOSITION 1. *For the MMI system (3), let* y_{k^-} *be the position (vector) realized at review time k before the renewal premium is collected; i.e.,* $y_{k^-} = y_{k-1} + \int_{k-1}^{k} dy_t$ *. Then there exists an optimal position* y^**, such that the optimal renewal premium is determined by the following base-stock policy:*

$$q_k = \begin{cases} y^* - y_{k^-} & \text{if } y_{k^-} < y^* \\ 0 & \text{otherwise} \end{cases}$$

We shall note that the optimal impulse feedback policy of the base-stock type described in Proposition 1 is obtained without regard to the type of information structure (i.e., unequal-risk versus equal-risk). Therefore, we shall confine our analysis of MMI systems to feedback policy, under either the unequal-risk or equal-risk information structure. For convenience, we use the term *unequal-risk* (or *equal-risk*) impulse feedback to differentiate base-stock feedback types under an *unequal-risk* (or *equal-risk*) information structure.

2.2. Heterogeneous Membership: Unequal-Risk Pooling

While complete information about positions and claim records is kept at the Club, each member *i* is only informed of the individual optimal impulse feedback position y^{i*}, and her own claims process x^i. The Club now needs to determine a risk level for the MMI system (3). For this purpose, we define an optimal unequal-risk risk level ξ^{i*} as:

$$\xi^{i*} \equiv \Pr\left(y^{i*} < x^i\right) \text{ for each } i$$

Given $x^i \sim N(\lambda_i, \sigma_i)$, we can then immediately write

$$y^{i*} = \lambda_i + \Phi^{-1}(1 - \xi^{i*})\sigma_i . \tag{5}$$

Under the unequal-risk impulse feedback, each member *i* is informed of y^{i*} and ξ^{i*}, which are kept confidential as its unequal-risk records. Note that an unequal-risk MMI system is in general unequal-risk pooling. At the Club level, an average risk level can be calculated using $\bar{\xi}^* = \frac{1}{n}\sum_{i=1}^{n} \xi^{i*}$ as an assessment of aggregate risk pooling. Let $\{y^*, \bar{\xi}^*\}$ denote the unequal-risk MI impulse feedback system, as described by Proposition 1.

3. Equal-Risk Pooling for Homogeneous Membership

Suppose that the managers of the same Club are compelled to operate with an equal-risk pooling scheme, under which an optimal base-stock policy, denoted by $\tilde{y}^{i*}$, is now open to every member of the Club. We shall note that the simplest implementation alternative of equal-risk pooling is to adopt homogeneous membership. Before we discuss the details of $\tilde{y}^* = (\tilde{y}^{1*}, \cdots, \tilde{y}^{n*})$, let us introduce the concept of equal-risk pooling regarding insurance claims. Let $\widetilde{Y}^* = \sum_{i=1}^{n} \widetilde{y}^{i*}$ be the total insurance position of the Club under an equal-risk information structure. In this case, the Club strives to maintain the aggregate position $\widetilde{Y}^*$ by collecting premiums from the members, which money is then used to collectively cover aggregate claims, $X = \sum_{i=1}^{n} x^i \sim N(\lambda,\ \sigma^2)$, where $\lambda = \sum_{i=1}^{n} \lambda_i$ and $\sigma = \sqrt{\sum_{i=1}^{n} \sigma_i^2}$. Because individual account records have been risk-equalized, the total minimized insurance position $\widetilde{Y}^*$ is expected to be reduced in comparison with the total unequal-risk position $Y^* = \sum_{i=1}^{n} y^{i*}$, where y^{i*} is the unequal-risk optimal position as given in Proposition 1. Thus, the allocation of total funds $\widetilde{Y}^*$ among heterogeneous members should no longer be determined based on an actual individual claims record x^i. Instead, the individual premium rates are determined according to the principle of information sharing; i.e., the principle of equal-risk sharing, as opposed to unequal-risk pooling as given in (5). The idea of the collective coverage of claims by the equal-risk allocation of premium contributions is termed *equal-risk pooling*.

3.1. Impulse Feedback Model under Equal-Risk Pooling

Thus, the result of equalizing risk pooling, (or equivalently homogenizing membership) is a potentially reduced effective share of claims coverage. The individual's effective share of claims coverage under equal-risk pooling, denoted as $\widetilde{x}^i \sim N(\widetilde{\lambda}_i, \widetilde{\sigma}_i^2)$, has the same mean as the actual individual claim x^i, but has a smaller variance (for $n > 1$) as a result of equal-risk pooling (or membership homogenizing); i.e.,

$$E(\widetilde{x}^i) = \widetilde{\lambda}_i = \lambda_i \text{, and } \operatorname{var}(\widetilde{x}^i) = \widetilde{\sigma}_i^2 < \sigma_i^2 . \qquad (6)$$

With the above characteristics, a member's effective share of the claims under equal-risk pooling is $\widetilde{x}^i \sim N(\lambda_i, \widetilde{\sigma}_i)$, as compared to its actual share of the claims $x^i \sim N(\lambda_i, \sigma_i)$. The effective and actual shares have the same mean λ_i, and only differ in standard deviation, specifically $\widetilde{\sigma}_i < \sigma_i$ for $n > 1$. Replacing actual σ_i with effective $\widetilde{\sigma}_i$, we can derive from equation (2) the dynamics for the effective position under equal-risk pooling, and express them in vector form as follows:

$$d\widetilde{y}_t = \mu(\widetilde{y}_t)dt + \widetilde{\Theta} dw_t ,$$

where $\widetilde{\Theta} = \left(\widetilde{\theta}_{ij}\right)_{n \times n}$ with $\widetilde{\theta}_{ii} = \widetilde{\sigma}_i$ and $\widetilde{\theta}_{ij} = 0$ for $i \neq j$. Then, with the same cost structure, we can derive from system (3) an equal-risk impulse feedback MMI system as being

$$\widetilde{\varphi}(y_0) = \min_{\widetilde{q}_k} E\left(\sum_{k \in T_q} K(\widetilde{q}_k) e^{-rk} + \int_0^T L(\widetilde{y}_t) e^{-rt} dt + \Omega(\widetilde{y}_T) e^{-rT} \right), \text{for any } y_0 \qquad (7)$$

subject to:

$$\begin{cases} d\widetilde{y}_t = \mu(\widetilde{y}_t)dt + \widetilde{\Theta} dw_t \\ \widetilde{y}_k = \widetilde{y}_{k-1} + \int_{k-1}^{k} d\widetilde{y}_t + \widetilde{q}_k \end{cases} .$$

The optimal equal-risk impulse feedback position $\widetilde{y}^* = (\widetilde{y}^{1*}, \cdots, \widetilde{y}^{n*})$ can then be obtained from the same QVI system (4), except for the operator $\widetilde{\mathsf{A}}$, which is modified with $\widetilde{\sigma}_i$ as:

$$\widetilde{\mathsf{A}} = \sum_{i=1}^{n} \tfrac{1}{2}\widetilde{\sigma}_i^2 \frac{\partial^2}{\partial(\widetilde{y}^i)^2}.$$

3.2. Equal-Risk Pooling under Impulse Feedback

In this subsection, we show that the optimal position $\widetilde{y}^* = (\widetilde{y}^{1*}, \cdots, \widetilde{y}^{n*})$ entails an equal-risk pooling scheme among all the members. For an equal-risk MMI system, an optimal aggregate risk level can be defined as

$$\widetilde{\xi}^* \equiv \Pr\left(\widetilde{Y}^* < X\right).$$

With the optimal aggregate risk level $\widetilde{\xi}^*$, an optimal equal-risk MMI impulse system can be denoted as $\left\{\widetilde{y}^*, \widetilde{\xi}^*\right\}$. An optimal equal-risk impulse feedback MMI system $\left\{\widetilde{y}^*, \widetilde{\xi}^*\right\}$ can be determined by solving the corresponding QVI system. An optimal unequal-risk impulse feedback MMI system was previously obtained as $\left\{y^*, \xi^*\right\}$.

Unlike the unequal-risk pooling MMI impulse system, we show below that individual impulse feedback under equal-risk pooling $\widetilde{y}^{i*}$, which is uniquely determined from the corresponding QVI system, can be implemented through an equal-risk level $\widetilde{\xi}^*$ for all members.

PROPOSITION 2. *Denote by* $\left\{\widetilde{y}^*, \widetilde{\xi}^*\right\}$ *an optimal equal-risk impulse feedback MMI system. Then, under an equal-risk pooling scheme at level* $\widetilde{\xi}^*$*, it holds that:*

$$\widetilde{\sigma}_i = \frac{\sigma_i^2}{\sigma} \quad \text{and} \quad \widetilde{y}^{i*} = \lambda_i + \Phi^{-1}(1-\widetilde{\xi}^*)\cdot\widetilde{\sigma}_i, \tag{8}$$

where $\widetilde{\sigma}_i$ *is the effective standard deviation under equal-risk pooling as given in (6).*

Proof. By the constraint that $\widetilde{\xi}^* = \Pr(\widetilde{Y}^* < X)$, we can write that:

$$\widetilde{Y}^* = \lambda + \Phi^{-1}(1-\widetilde{\xi}^*)\sigma,$$

or, equivalently:

$$\sum_{i=1}^{n}\left(\lambda_i + \Phi^{-1}(1-\widetilde{\xi}^*)\frac{\sigma_i^2}{\sigma} - \widetilde{y}^{i*}\right) = 0.$$

Letting each item of the summation above be zero, we obtain:

$$\widetilde{y}^{i*} = \lambda_i + \Phi^{-1}(1-\widetilde{\xi}^*)\frac{\sigma_i^2}{\sigma}.$$

Letting $\widetilde{\sigma}_i = \dfrac{\sigma_i^2}{\sigma}$, we can write the above equivalently as follows:

$$\Phi(1-\widetilde{\xi}^*) = \Pr(\widetilde{x}^i \le \widetilde{y}^{i*}),$$

where $\tilde{x}^i \sim N(\lambda_i, \tilde{\sigma}_i^2)$. This concludes the proof of Proposition 2.

4. The Value of Information Sharing

4.1. Define the Value of Information sharing

We have thus far obtained two value functions associated respectively with two risk pooling structures of an MMI system, namely, the value function φ of an unequal-risk system $\{y^*, \overline{\xi}^*\}$ versus the value function $\widetilde{\varphi}$ of an equal-risk system $\{\widetilde{y}^*, \widetilde{\xi}^*\}$. By definition, a value function represents the total value (a cost in this case) of a Club optimally operated under a specific risk pooling structure, either unequal-risk pooling or equal-risk pooling. In this sense, the difference between the two value functions can be used as a measure of the value (or the prize) for information sharing, especially the difference in the two value functions when evaluated at their respective optimal base-stock positions. For this purpose, we define the value of information sharing as:

$$P_{\text{mutual}} \equiv \varphi(y^*) - \widetilde{\varphi}(\widetilde{y}^*), \tag{8*}$$

where y^* and $\widetilde{y}^*$ are the optimal base-stock positions respectively under unequal- and equal-risk.

First, we need to be assured that a nonnegative difference between the two value functions exists (i.e., $\varphi - \widetilde{\varphi} > 0$, and therefore $P_{\text{mutual}} = \varphi^* - \widetilde{\varphi}^* > 0$). Intuitively, the difference is nonnegative (i.e.,), since equal-risk pooling should reduce the total insurance cost. Now let us ascertain this intuition using rigorous analysis. Since $\widetilde{\sigma}_i < \sigma_i$ for $n > 1$, we can write $\sigma_i = \widetilde{\sigma}_i + \Delta\sigma_i$ for some $\Delta\sigma_i > 0$. Then, the HJB equation of an unequal-risk $\{y^*, \overline{\xi}^*\}$ can be derived from system (4) as follows:

$$r\varphi - \left(\sum_{i=1}^{n} \tfrac{1}{2}\left(\widetilde{\sigma}_i^2 + (\Delta\sigma_i)^2\right) \frac{\partial^2}{\partial y_i^2} \right)\varphi - H(y, D\varphi) = 0\,. \tag{9}$$

Or, equivalently, we can write the above as:

$$\widetilde{F}(y, \varphi, D\varphi, \mathsf{A}\varphi) - \left(\sum_{i=1}^{n} \tfrac{1}{2}(\Delta\sigma_i)^2 \frac{\partial^2}{\partial y_i^2} \right)\varphi = 0\,,$$

where $\widetilde{F}(\cdot) = r\varphi - \left(\sum_{i=1}^{n} \tfrac{1}{2}\widetilde{\sigma}_i^2 \frac{\partial^2}{\partial y_i^2} \right)\varphi - H(y, D\varphi)$ is an augmented Hamiltonian for the equal-risk MMI system $\{\widetilde{y}^*, \widetilde{\xi}^*\}$, of which the HJB equation can be written as:

$$\widetilde{F}(y, \widetilde{\varphi}, D\widetilde{\varphi}, \mathsf{A}\widetilde{\varphi}) = 0\,.$$

With $\dfrac{\partial^2\varphi}{\partial y_i^2} > 0$ and $\Delta\sigma_i > 0$ for some *i*, the HJB equation (9) of an unequal-risk MMI system can immediately verify that

$$\widetilde{F}(y, \varphi, D\varphi, \mathsf{A}\varphi) > \widetilde{F}(y, \widetilde{\varphi}, D\widetilde{\varphi}, \mathsf{A}\widetilde{\varphi})\,.$$

The fact that the Hamiltonian *H* is identical for both unequal-risk and equal-risk systems implies that φ and $\widetilde{\varphi}$ can only differ by an additive functional term; i.e., $\varphi = \widetilde{\varphi} + \delta$ for some functional term $\delta \geq 0$. With this, the value of information sharing can be analytically measured by δ. In principle, the exact expression of this price term δ can be obtained by solving for φ and $\widetilde{\varphi}$ from the respective HJB equations of the unequal-risk MMI system $\{y^*, \overline{\xi}^*\}$ and the equal-risk MMI system $\{\widetilde{y}^*, \widetilde{\xi}^*\}$, respectively. However, closed-form solutions are often unattainable, and even numerical solutions are still too complex to be tractable using numerical methods. In the Proposition below, we derive a more tractable lower-bound function $\underline{\delta}$, which gives the least cost difference caused by information privacy.

PROPOSITION 3. *Let φ and $\widetilde{\varphi}$ be the respective value functions under the unequal-risk and equal-risk MMI systems with a strictly convex Lagrangian L. Then, for each non-impulse interval*

$[k, k+1)$ *for all* $k \in T_q$, *the following holds true:*

1) *The value functions* φ *and* $\widetilde{\varphi}$ *are strictly convex for* $t \in [k, k+1)$ *; i.e.,* $\frac{\partial^2 \varphi}{\partial y_i^2} > 0$ *and* $\frac{\partial^2 \widetilde{\varphi}}{\partial y_i^2} > 0$ *for all i.*
2) *There exists a functional* $\delta > 0$, *such that* $\varphi = \widetilde{\varphi} + \delta$. *Therefore,* $P_{\text{privacy}} = \delta^* > 0$, *where* $\delta^* = \varphi(y^*) - \widetilde{\varphi}(\widetilde{y}^*)$.
3) *Given* φ *and* $\widetilde{\varphi}$, *then* $\delta = \varphi - \widetilde{\varphi}$ *has a nonnegative Hamiltonian, specifically,*

$$F(y, \delta, D\delta, \mathsf{A}\delta) = \left(\sum_{i=1}^{n} \tfrac{1}{2} (\Delta\sigma_i)^2 \frac{\partial^2}{\partial y_i^2} \right) \widetilde{\varphi} > 0,$$

where $\Delta\sigma_i \equiv \sigma_i - \widetilde{\sigma}_i$.

Proof. Item 1 above is a proven result in control theory (Fleming and Soner, 2006), and its proof is thus omitted. Given $\frac{\partial^2 \varphi}{\partial y_i^2} > 0$ for all *i*, item 2 of Proposition 3 can be verified from the HJB equation (9). To prove item 3 of the Proposition, we obtain from (9) the following variational inequality:

$$F(y, \varphi, D\varphi, \mathsf{A}\varphi) > \widetilde{F}(y, \varphi, D\varphi, \widetilde{\mathsf{A}}\varphi),$$

Where $F(\cdot)$ differs from $\widetilde{F}(\cdot)$ only in the second-order operator A. Then, using $\varphi = \widetilde{\varphi} + \delta$, we can verify from HJB (9) that

$$F(y, \widetilde{\varphi}, D\widetilde{\varphi}, \mathsf{A}\widetilde{\varphi}) + F(y, \delta, D\delta, \mathsf{A}\delta) = 0.$$

Noting that $\widetilde{F}(y, \widetilde{\varphi}, D\widetilde{\varphi}, \widetilde{\mathsf{A}}\widetilde{\varphi}) = 0$ and $\mathsf{A} = \widetilde{\mathsf{A}} + \sum_{i=1}^{n} \tfrac{1}{2} (\Delta\sigma_i)^2 \frac{\partial^2}{\partial y_i^2}$, we can write the above HJB equation as follows:

$$F(y, \delta, D\delta, \mathsf{A}\delta) - \left(\sum_{i=1}^{n} \tfrac{1}{2} (\Delta\sigma_i)^2 \frac{\partial^2}{\partial y_i^2} \right) \widetilde{\varphi} = 0.$$

Noting that $\frac{\partial^2 \widetilde{\varphi}}{\partial y_i^2} > 0$ and $\Delta\sigma_i > 0$ for $n > 1$, the proof of item 3 of Proposition 3 is immediate from the above equality. With this, we conclude the proof.

4.2. Computing the Value of Information Sharing

In summary, the exact value of information sharing, denoted by $V_{\text{mutual}} \equiv \varphi(y^*) - \widetilde{\varphi}(\widetilde{y}^*)$, can be computed as follows:

1) Solve the unequal-risk and equal-risk HJB equations respectively for φ and $\widetilde{\varphi}$.
2) Obtain the respective optimal base-stock positions, y^* (unequal-risk) and $\widetilde{y}^*$ (equal-risk).
3) Then compute the value of information sharing, i.e., $V_{\text{mutual}} = \varphi(y^*) - \widetilde{\varphi}(\widetilde{y}^*)$.

The above solution for the exact price of privacy requires the solving of two HJB equations, one for the unequal-risk system and the other for the equal-risk system. These equations mostly require complex numerical methods to solve them. However, using Proposition 3, we can construct an approximate solution, which requires much less computation. The scheme for the approximate method of solution is described below:

1) Solve for the equal-risk $\tilde{\varphi}$ for its HJB equation, and obtain $\tilde{y}^*$.
2) Determine:

$$\Delta V^* = \left(\sum_{i=1}^{n} \tfrac{1}{2} (\Delta\sigma_i)^2 \frac{\partial^2}{\partial y_i^2} \right) \tilde{\varphi}(\tilde{y}^*) .$$

3) Determine a linear functional difference $\hat{\delta} \in C^1$ with $D^2 \hat{\delta} = 0$ by solving the following first-order PDE system:

$$r\hat{\delta} - H(y, D\hat{\delta}) - \Delta V^* = 0 \text{, subject to: } \frac{\partial^2 \hat{\delta}}{\partial y_i^2} = 0 \text{ for all } i.$$

4) Then compute the approximate value of information sharing $\hat{V}_{\text{mutual}} = \hat{\delta}(\tilde{y}^*)$.

The rationale of the above approximation is to seek the approximate difference in the form of a linear function (i.e., $D^2 \hat{\delta} = 0$), so as to avoid solving the HJB equations twice.

4.3. Value of Information Sharing and Volatility of Risk

Compared with the computation of the value of information sharing, a more important and interesting topic concerns the characteristics of the value of information sharing, i.e., $V_{\text{mutual}} = \varphi(y^*) - \tilde{\varphi}(\tilde{y}^*)$. By Item 3 of Proposition 3, the characteristics of the information sharing value are characterized in the corresponding Hamiltonian $F(y, \delta, D\delta, \mathsf{A}\delta)$, which has been obtained in Proposition 3 as:

$$F(y, \delta, D\delta, \mathsf{A}\delta) = \left(\sum_{i=1}^{n} \tfrac{1}{2} (\Delta\sigma_i)^2 \frac{\partial^2}{\partial y_i^2} \right) \tilde{\varphi} > 0 ,$$

where $\Delta\sigma_i \equiv \sigma_i - \tilde{\sigma}_i$ is the variability differential between non-risk pooling and equal-risk pooling. For the 'ideal' case of homogeneous membership with i.i.d. individual claim processes with identical $\sigma_i = \bar{\sigma}$ for all $i = 1, \cdots, n$, the variability differential can be determined as $\Delta\sigma_i = \sigma_i - \tilde{\sigma}_i = (1 - \frac{1}{\sqrt{n}})\bar{\sigma}$. In terms of the Hamiltonian characterization of $V_{\text{mutual}} = \varphi(y^*) - \tilde{\varphi}(\tilde{y}^*)$, it can immediately be seen that the value of information sharing increases along with the average variability $\bar{\sigma}$ and the size of the Club *n*. Both parameters $\bar{\sigma}$ and *n* are measures or indicators of the volatility of underlying risks in terms of aggregate claims. With the finding in this paper that homogeneity facilitates optimal realization of the value of information sharing, it is without loss of generality to conclude that the more volatile the insurance risk is, the more competitive the mutual insurance becomes.

5. Findings and Implementation

First, let us summarize the useful observations and managerial implications that can be drawn from the results we have obtained so far in this paper, especially from Proposition 5.

5.1. Findings and Implications

1) The price of information privacy is mainly dependent on $\Delta\sigma_i = \sigma_i - \tilde{\sigma}_i$, where $\tilde{\sigma}_i = \frac{\sigma_i^2}{\sigma}$ and $\sigma = \sqrt{\sum_{i=1}^{n} \sigma_i^2}$. If the claims are deterministic ($\Delta\sigma_i = 0$), then the price of privacy would vanish, that is, there would be no difference between an unequal-risk and an equal-risk information structure without regard to the degree of heterogeneity of the members.
2) A unified tonnage-based premium policy can be justified only if the tonnage of a vessel is linearly

associated with the variability of the claims incurred by the vessel. This finding suggests that a rigorous and intensive statistical study of the correlation between tonnage and its claims needs to be conducted, so as to determine whether or not a tonnage-based premium policy is justifiable.

3) Under an unequal-risk information structure, the total cost minimizing premium policy entails an unequal-risk pooling scheme among the heterogeneous members of a P&I Club. Under the optimal unequal-risk premium policy, the individual's share of risk is solely determined by its actual claims record.
4) Given the fact that the tonnage-based premium policy has been practiced under unequal-risk information in P&I Clubs since their establishment 150 years ago, let us suppose that the tonnage-base premium policy is justified (i.e., that tonnage is linearly associated with the variance of claims). The above findings 1), 2), and 3) then explain the phenomenon in marine insurance that vessels of a similar tonnage tend to join the same P&I Club.

5.2. Implementation of Equal-Risk Pooling

The principle of information sharing implies the equal-sharing of risks. Noting that the equal-pooling of risk can be attained with an optimal premium policy under an equal-risk information structure, we only need to examine how to implement an unequal-risk equal-risk pooling scheme; i.e., equal-risk pooling in an unequal-risk MMI system $\{y^*, \bar{\xi}^*\}$. Two scenarios for implementing equal-risk pooling in an unequal-risk MMI system can be immediately considered: One with an equal-average level of risk $\bar{\xi}^* = \frac{1}{n}\sum_{i=1}^{n} \xi^{i*}$, where ξ^{i*} is as that determined in (5), and the other with an equal-level of risk $\tilde{\xi}^*$. In what follows, we examine the details of the implementation of unequal-risk equal-risk pooling.

The insurance threshold position for each member i under the equal-average risk $\bar{\xi}^*$ scheme, denoted by $\bar{y}^i(\bar{\xi}^*)$, can be determined to be:
$\bar{y}^i(\bar{\xi}^*) = \lambda_i + \Phi^{-1}(1-\bar{\xi}^*)\sigma_i$.

In vector form, we write the equal-average risk pooling position as $\bar{y}(\bar{\xi}^*) = (\bar{y}^1(\bar{\xi}^*), \cdots, \bar{y}^n(\bar{\xi}^*))$. From equation (5), we can determine that in general the equal-average risk pooling position $\bar{y}(\bar{\xi}^*)$ and the optimal unequal-risk pooling position y^* differ (i.e., $\bar{y}(\bar{\xi}^*) \neq y^*$). Individually, some would have a higher threshold position (i.e., $\bar{y}^i(\bar{\xi}^*) > y^{i*}$) if $\bar{\xi}^* < \xi^{i*}$, and some may have a lower threshold position (i.e., $\bar{y}^i(\bar{\xi}^*) < y^{i*}$) if $\bar{\xi}^* > \xi^{i*}$.

Since the total insurance cost is minimized under an optimal threshold y^*, we can conclude that the total cost under an equal-average risk scheme, denoted by $\varphi(\bar{y})$, can be no less than the unequal-risk value function $\varphi(y^*)$ (i.e., $\varphi(\bar{y}(\bar{\xi}^*)) \geq \varphi(y^*)$). This suggests that, in terms of total Club cost, under an unequal-risk MMI system it is worse to implement an equal-average risk pooling scheme than it is to implement an optimal unequal-risk pooling scheme.

Now let us examine what happens when an equal-equal-risk risk level $\tilde{\xi}^*$ is implemented for the unequal-risk MMI system. In this case, the individual threshold position, denoted by $\bar{y}^i(\tilde{\xi}^*)$, can be determined as:
$\bar{y}^i(\tilde{\xi}^*) = \lambda_i + \Phi^{-1}(1-\tilde{\xi}^*)\sigma_i$.

Using similar arguments, we can show that $\varphi(\bar{y}(\tilde{\xi}^*)) \geq \varphi(y^*)$. Thus, we can conclude that in terms of total insurance cost, an unequal-risk pooling system is better for the allocation of premiums in an of total insurance cost, an unequal-risk pooling system is better for the allocation of premiums in an unequal-risk MMI system.

6. Conclusion

It is concluded that an open policy or equal-risk information can lead to a more efficient MMI system overall for society, and to a greater degree of fairness and information sharing for the insured. The paper determines that information sharing can achieve best social welfare as well as efficient operation of a P&I Club. The study provides a scientific basis for future legislation on MMI competition law. The conclusion provides a scientific basis for both managerial strategy and competition regulation.

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